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BOCHNER'S THEOREM IN INFINITE DIMENSIONS

by

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BOCHNER'S THEOREM IN INFINITE DIMENSIONS

1. Introduction

Let G be a locally compact abelian group. A well-known theorem of Bochner ([1], [2]) states that a mapping ψ of G into C is positive definite and continuous if and only if there is a unique nonnegative finite regular Borel measure m_ψ on $\hat{G}$ (the dual group of G) such that $\psi(g) = \int_{\hat{G}}(\gamma, g)dm_\psi$ where (γ, g) denotes the action of the character γ on g. An alternate version of the theorem ([3]) states that if A is a semi-simple, self-adjoint, commutative Banach algebra and ψ is a linear functional on A, then ψ is positive and extendable if and only if there is a finite positive Baire measure ν_ψ on $\mathcal{M}$ (the maximal ideal space of A) such that $\psi(\alpha) = \int_{\mathcal{M}}\hat{\alpha}(M)d\nu_\psi$ where $\hat{\alpha}$ is the Gelfand transform of $\alpha \in A$. Here we shall extend these theorems to mappings taking values in a Banach space X. Our results generalize the extension of Bochner's theorem made in [4].

We shall, in fact, first prove that if A is a self-adjoint, commutative Banach algebra and ψ is a linear map of A into the Banach space X, then ψ is positive+ and "almost" extendable if and only if there is a weak-*-regular, finite, positive set function ν_ψ^{**} mapping $\Sigma(\mathcal{M})$ (the Borel field of $\mathcal{M}$) into X^{**} such that $\psi(\alpha) = \int_{\mathcal{M}}\hat{\alpha}(M)d\nu_\psi^{**}$ (where $\psi(\alpha)$ is viewed as an element of X^{**}). We next show that if the mapping $\hat{\psi}$ of $\hat{A}$ into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$ is weakly compact++, then ν_ψ^{**} can be viewed as a weakly regular positive vector measure ν_ψ

+Positivity is with respect to a suitable cone in X.

++This means that $\hat{\psi}$ maps bounded sets in A into weakly compact sets in X.

mapping $\Sigma(\mathscr{M})$ into X and, conversely, if $\psi(\alpha) = \int_{\mathscr{M}} \hat{\alpha}(M)d\nu_{\psi}$ where ν_{ψ} is a weakly regular positive vector measure on $\Sigma(\mathscr{M})$ to X, then ψ is positive and "almost" extendable and $\hat{\psi}$ is weakly compact. In the case where $A = L_1(G,C)$, these results lead to a representation of ψ by an element p_{ψ} of $L_{\infty}(G,X)$ i.e. $\psi(\alpha) = \int_G \alpha(g)p_{\psi}(g)d\mu$ where μ is the Haar measure on G. We then develop an extended Bochner's theorem for maps p in $L_{\infty}(G,X)$. Finally, we use some particular Banach spaces to illustrate the theory.

The general results obtained here are combined with the transform theory on $L_1(G,X)$ to develop an inversion theorem and a Plancherel theorem in [5]. These theorems are also applied to the solution of convolution equations in Hilbert spaces in [5]. The convolution equations arise in the study of problems relating to the stability and control of systems described by parabolic partial differential equations.

2. Positive Functions

Let X be a Banach space and let X^* and X^{**} be the dual spaces of X and X^*, respectively. If φ is an element of X^*, then the operation of φ on x is denoted by (x,φ). The notion of positivity that we use is based on a cone of "positive" elements contained in X. We assume that the cone is defined by a family of elements of X^*. More precisely, we have

DEFINITION 2.1. *Let Φ be a subset of X^*. The subset K_{Φ} (or simply K when Φ is fixed by the context) of X given by*

(2.2) $$K_{\Phi} = \{x \in X\colon (x,\varphi) \geq 0 \ \text{for all} \ \varphi \ \text{in} \ \Phi\}$$

is called the cone determined by Φ.

Now let A be a Banach algebra with an involution given by $\alpha \to \alpha^*$, $\alpha \in A$, and let ψ be a linear mapping of A into X. We then have

DEFINITION 2.3. *The mapping* ψ *is positive with respect to the cone* K_Φ *(or* Φ*-positive) if* $\psi(\alpha\alpha^*) \in K_\Phi$ *for all* α *in* A.

We observe that ψ is Φ-positive if and only if the mappings $(\psi(\cdot),\varphi)$ of A into C are positive functionals for all φ in Φ. Note also that if ψ is Φ-positive, then, for any φ in Φ, the functional $B_\varphi(\alpha,\beta)$ given by

$$B_\varphi(\alpha,\beta) = (\psi(\alpha\beta^*),\varphi) \tag{2.4}$$

is a symmetric bilinear form satisfying the Cauchy inequality

$$|B_\varphi(\alpha,\beta)|^2 \leq B_\varphi(\alpha,\alpha)B_\varphi(\beta,\beta) \tag{2.5}$$

for α,β in A.

DEFINITION 2.6. *The mapping* ψ *is symmetric with respect to* Φ *(or simply symmetric) if* $(\psi(\alpha),\varphi) = \overline{(\psi(\alpha^*),\varphi)}$ *for all* φ *in* Φ *and* α *in* A.

If A has a unit e, then every Φ-positive mapping is symmetric since $(\psi(\alpha),\varphi) = (\psi(\alpha e),\varphi) = B_\varphi(\alpha,e) = \overline{B_\varphi(e,\alpha)} = \overline{(\psi(\alpha^*),\varphi)}$ for all φ. If A does not have a unit, then A can be imbedded in an algebra $\tilde{A} = A \oplus C$ with a unit in a natural way. Letting e be the unit in $\tilde{A}$, we can extend

ψ to a linear mapping $\tilde{\psi}_{x_o}$ of $\tilde{A}$ into X by setting $\tilde{\psi}_{x_o}(\alpha+ce) = \psi(\alpha)+cx_o$ for a given x_o in X. Clearly ψ is symmetric if and only if $\tilde{\psi}_{x_o}$ is. We now have

DEFINITION 2.7. _A Φ-positive mapping ψ is almost extendable if_ (i) ψ _is symmetric,_ (ii) ψ _is continuous, and_ (iii) $|(\psi(\alpha),\varphi)|^2 \leq d\|\psi\|\|\varphi\|(\psi(\alpha\alpha^*),\varphi)$ _for all_ φ _in_ Φ _and_ α _in_ A _where_ d _is a constant with_ $d \geq 1$.

DEFINITION 2.8. _A Φ-positive mapping ψ is extendable if ψ is symmetric and if there is an_ x_o _in_ X _such that_ $|(\psi(\alpha),\varphi)|^2 \leq (x_o,\varphi)(\psi(\alpha\alpha^*),\varphi)$ _for all_ φ _in_ Φ _and_ α _in_ A.

If A has a unit e, then any Φ-positive mapping is extendable. If A does not have a unit, then we have

PROPOSITION 2.9. _A Φ-positive mapping ψ is extendable if and only if there is an extension $\tilde{\psi}$ of ψ to $\tilde{A}$ which is Φ-positive._

Proof: If $\tilde{\psi}$ is a Φ-positive extension of ψ and e is the unit in $\tilde{A}$, then, letting $x_o = \tilde{\psi}(e)$, we deduce immediately that $|(\psi(\alpha),\varphi)|^2 = |(\tilde{\psi}(\alpha),\varphi)|^2 \leq (x_o,\varphi)(\tilde{\psi}(\alpha\alpha^*),\varphi) = (x_o,\varphi)(\psi(\alpha\alpha^*),\varphi)$ (by 2.5) and that ψ is symmetric.

On the other hand, if ψ is extendable, then let $\tilde{\psi}(\alpha+ce) = \tilde{\psi}_{x_o}(\alpha+ce) = \psi(\alpha)+cx_o$. Since $(\tilde{\psi}([\alpha+ce][\alpha+ce]^*),\varphi) = (\psi(\alpha\alpha^*),\varphi)+2\mathrm{Re}\,\bar{c}(\psi(\alpha),\varphi) + |c|^2(x_o,\varphi)$, we have $(\tilde{\psi}([\alpha+ce][\alpha+ce]^*),\varphi) \geq (\psi(\alpha\alpha^*),\varphi)-2|c||(\psi(\alpha),\varphi)| + |c|^2(x_o,\varphi) \geq [(\psi(\alpha\alpha^*),\varphi)^{1/2} - |c|(x_o,\varphi)^{1/2}]^2 \geq 0$ (as ψ is extendable). Thus, $\tilde{\psi}$ is Φ-positive.

PROPOSITION 2.10. _If there is an approximate identity_ $\{e_n\}$ _in_ A, _then a continuous Φ-positive mapping ψ is almost extendable._

Proof: Since $(\psi(\alpha^*),\varphi) = \lim_{n\to\infty}(\psi(e_n\alpha^*),\varphi) = \lim_{n\to\infty}\overline{(\psi(\alpha e_n^*),\varphi)} = \overline{(\psi(\alpha),\varphi)}$, ψ is symmetric and, since $|B_\varphi(e_n,\alpha)|^2 \leq B_\varphi(e_n,e_n)B_\varphi(\alpha,\alpha) \leq \|\psi\|\|\varphi\|B_\varphi(\alpha,\alpha) = \|\psi\|\|\varphi\|(\psi(\alpha\alpha^*),\varphi)$, ψ is almost extendable.

In order to prove the extension of Bochner's theorem, we require a condition on the family Φ defining the cone of "positive" elements. As we shall see, the essential point is to deduce an estimate of the form $\|\psi(\alpha)\|^2 \leq k\|\psi(\alpha\alpha^*)\|$ from estimates of the form $|(\psi(\alpha),\varphi)|^2 \leq d\|\varphi\|^2\|\psi\|\|\psi(\alpha\alpha^*)\|$ (ψ almost extendable) or $\|(\psi(\alpha),\varphi)|^2 \leq \|\varphi\|^2\|x_0\|\|\psi(\alpha\alpha^*)\|$ (ψ extendable). The following definition allows us to do this.

DEFINITION 2.11. <u>The family Φ is full if there is a $\rho > 0$ such that</u>

(2.12) $$\|x\| \leq \rho \sup_{\substack{\varphi\in\Phi \\ \varphi\neq 0}} \{|(x,\varphi)|/\|\varphi\|\}$$

<u>for all x in X.</u>[+]

We now have

LEMMA 2.13. <u>If A has a unit e, if the involution on A is continuous, and if Φ is full, then every Φ-positive mapping ψ is continuous and almost extendable.</u>

Proof: Suppose first that α is a Hermitian element of A with $\|\alpha\| \leq 1$. The binomial series $(1-t)^{1/2} = 1 - \frac{t}{2} - \frac{t^2}{2^2 2!} - \dots$ converges absolutely for

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[+]This could be replaced by the following: Φ is full relative to ψ if there is a $\rho > 0$ such that $\|\psi(\alpha)\| \leq \rho \sup_{\substack{\varphi\in\Phi \\ \varphi\neq 0}} \{|(\psi(\alpha),\varphi)|/\|\varphi\|\}$ for all α in A.

$|t| \leq 1$ and so the series $e - \frac{\alpha}{2} - \frac{\alpha^2}{2^2 2!} - \dots$ converges absolutely in A. Since the involution is continuous, the sum β of this series is a Hermitian element of A with $\beta\beta^* = \beta^2 = e-\alpha$. It follows that $(\psi(e-\alpha),\varphi) = (\psi(\beta\beta^*),\varphi) \geq 0$ and hence, that $(\psi(e),\varphi) \geq (\psi(\alpha),\varphi)$. Replacing α by $-\alpha$, we have $(\psi(e),\varphi) \geq (\psi(-\alpha),\varphi)$. But $(\psi(\alpha),\varphi)$ is real (since α is Hermitian) and so $|(\psi(\alpha),\varphi)| \leq \|\varphi\|\|\psi(e)\|$. Since Φ is full, $\|\psi(\alpha)\| \leq \rho\|\psi(e)\|$.

Now, if α is any element of A, then $\alpha = \frac{1}{2}(\alpha+\alpha^*) - \frac{i}{2}(i(\alpha-\alpha^*))$. Since the involution is continuous, there is a $c > 0$ such that $\|\alpha^*\| \leq c\|\alpha\|$ and so, if $\|\alpha\| \leq 2/c+1$, then $\|(\alpha+\alpha^*)/2\| \leq 1$ and $\|i(\alpha-\alpha^*)/2\| \leq 1$. It follows that $\|\psi(\alpha)\|^2 \leq 2\rho\|\psi(e)\|$ for all α in A with $\|\alpha\| \leq 2/c+1$. Thus, ψ is bounded and therefore continuous.

Since $|(\psi(\alpha),\varphi)|^2 \leq (\psi(e),\varphi)(\psi(\alpha\alpha^*),\varphi) \leq \|\psi\|\|\varphi\|(\psi(\alpha\alpha^*),\varphi)$, ψ is almost extendable.

COROLLARY 2.14. _If the involution on A is continuous, if Φ is full, and if ψ is Φ-positive and extendable, then ψ is continuous and almost extendable._

Proof: Apply proposition 2.9 and the lemma.

Let G be a σ-finite locally compact abelian group and let $A = L_1(G,C)$. The involution on $L_1(G,C)$ is given by $\alpha^*(g) = \overline{\alpha(-g)}$ and is continuous since $L_1(G,C)$ is semi-simple. Observe that if Φ is full and ψ is a Φ-positive mapping of $L_1(G,C)$ into X, then ψ is continuous and almost extendable if ψ is extendable (corollary 2.14) and conversely, ψ is almost extendable if ψ is continuous (proposition 2.10).

Now let us introduce the following

DEFINITION 2.15 *Let* p *be an element of* $L_\infty(G,X)$. *The mapping* p *is* Φ-*positive definite if*

$$(2.16)\qquad \sum_{n=1}^{N}\sum_{m=1}^{N} c_n\bar{c}_m(p(g_n-g_m),\varphi) \geq 0$$

for any integer N, *any* $c_1,\ldots,c_N$ *in* C, *any* $g_1,\ldots,g_N$ *in* G, *and all* φ *in* Φ. *The mapping* p *is integrally* Φ-*positive definite if*

$$(2.17)\qquad \left(\int_G\int_G \alpha(g)\overline{\alpha(g')}p(g-g')\,d\mu d\mu,\varphi\right) \geq 0$$

for all α *in* $L_1(G,C)$ *and all* φ *in* Φ.

We then have

PROPOSITION 2.18. *Let* p *be a continuous element of* $L_\infty(G,X)$. *Then* p *is* Φ-*positive definite if and only if* p *is integrally* Φ-*positive definite.*

Proof: If p is Φ-positive definite, then p is integrally Φ-positive definite by a result of Naimark ([6], p. 397). Conversely, if p is integrally Φ-positive definite, then there is a continuous positive definite function f_φ mapping G into C such that $f_\varphi(g) = (p(g),\varphi)$ locally almost everywhere on G ([6], p. 397) for each φ in Φ. Since $(p(\cdot),\varphi)$ is continuous, $f_\varphi(\cdot) = (p(\cdot),\varphi)$ everywhere and hence, p is Φ-positive definite.

Now it is a fact that ψ is a bounded weakly compact linear map of $L_1(G,C)$ into X with separable range if and only if there is a p with (essentially) weakly compact range in $L_\infty(G,X)$ such that

$$(2.19) \qquad \psi(\alpha) = \int_G \alpha(g)p(g)d\mu$$

for all α in $L_1(G,C)$ ([9], p. 279, or [7], p. 507). Moreover, $\|\psi\| = \|p\|_\infty$. The fact that the weakly compact maps in $\mathcal{L}(L_1(G,C),X)$ are essentially the same as the functions with (essentially) weakly compact range in $L_\infty(G,X)$ will allow us to relate the notion of Φ-positivity to the notions of Φ-positive definiteness and integral Φ-positive definiteness.

LEMMA 2.20. *Let Φ be full. If ψ is a weakly compact linear mapping of $L_1(G,C)$ into X which is Φ-positive and extendable, then there is an (essentially unique) integrally Φ-positive p in $L_\infty(G,X)$ such that*

$$(2.21) \qquad \psi(\alpha) = \int_G \alpha(g)p(g)d\mu$$

for all α in $L_1(G,C)$. Conversely, if p is an integrally Φ-positive definite element of $L_\infty(G,X)$ and ψ is given by 2.21, then ψ is Φ-positive and almost extendable.

Proof: Assume that ψ is given. In view of [9], p. 279, the mapping p exists and we need only show that p is integrally Φ-positive definite. But

$$(2.22) \qquad \psi(\alpha\alpha^*) = \int_G\int_G \alpha(g-g')\overline{\alpha(-g')}p(g)d\mu d\mu = \int_G\int_G \alpha(g)\overline{\alpha(g')}p(g-g')d\mu d\mu$$

by virtue of the Fubini and Tonelli theorems and the invariance of Haar measure. Conversely, given p, we simply note that $\psi(\alpha\alpha^*)$ is determined by 2.22 in order to prove that ψ is Φ-positive. Moreover, since ψ is continuous, ψ is almost extendable by proposition 2.10.

3. Bochner's Theorem for Algebras

Before proving the generalization of Bochner's theorem to maps of A into X, we recall the following

DEFINITION 3.1. *Let S be a locally compact topological space and let $\Sigma(S)$ be the Borel field of S. A vector measure ν is a weakly countably additive set function taking values in X. The vector measure ν is weakly regular if the scalar measures $(\nu(\cdot),\varphi)$ are regular*+ *for all φ in X^*. The vector measure ν is Φ-positive if $(\nu(E),\varphi) \geq 0$ for all φ in Φ and E in $\Sigma(S)$. A set function ν^{**} mapping $\Sigma(S)$ into X^{**} is weak-*-regular if $(\varphi,\nu^{**}(\cdot))$ is a regular scalar measure for all φ in X^*. The set function ν^{**} is Φ-positive if $(\varphi,\nu^{**}(E)) \geq 0$ for all φ in Φ and E in $\Sigma(S)$.*

We now have

THEOREM 3.2. *Let A be a self-adjoint commutative Banach algebra whose involution satisfies the condition $(\widehat{a^*}) = \bar{\hat{a}}$ (e.g. A semi-simple) and let Φ be a full family. If ψ is a mapping of A into X, then ψ is Φ-positive and almost extendable if and only if there is a set function ν^{**} mapping $\Sigma(\mathcal{M})$ into X^{**} such that* (i) *ν^{**} is weak-*-regular,* (ii) *ν^{**} is Φ-positive,* (iii) *ν^{**} is finite i.e. $\|\nu^{**}\|(\mathcal{M}) < \infty$,* (iv) *the mapping $T_{\nu^{**}}$*

+ A scalar measure μ is regular if given $\epsilon > 0$ and $E \in \Sigma(S)$ with $\|\mu\|(E) < \infty$, then there is a compact $K \subseteq E$ and an open $O \supseteq E$ such that $\|\mu\|(O-K) < \epsilon$.

of X^* *into the scalar measures on* $\mathcal{M}$ *given by* $T_{\nu^{**}}(\varphi) = (\nu^{**}(\cdot),\varphi)$ *is continuous in the* X *and* $C_o(\mathcal{M})^+$ *topologies in these spaces respectively, and* (v)

$$(\psi(\alpha),\varphi) = \int_{\mathcal{M}} \hat{\alpha}(M)\,d(\nu^{**},\varphi) \tag{3.3}$$

for all α *in* A *and all* φ *in* X^*.

Proof: Suppose first that ψ is Φ-positive and almost extendable. Then ψ is continuous. Let $\hat{\psi}$ be the map of $\hat{A}$ into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$. Then $\|\hat{\psi}(\hat{\alpha})\| = \|\psi(\alpha)\|$ and $|(\psi(\alpha),\varphi)|^2 \leq d\|\psi\|\|\varphi\|(\psi(\alpha\alpha^*),\varphi) \leq d\|\psi\|\|\varphi\|^2\|\psi(\alpha\alpha^*)\|$ for all φ in Φ (since ψ is almost extendable). Since Φ is full, there is a $\rho > 0$ such that $\|\psi(\alpha)\| \leq \rho \sup\limits_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(\psi(\alpha),\varphi)|/\|\varphi\|\}$. Thus, there is a constant k $(= \rho^2 d\|\psi\|)$ such that

$$\|\psi(\alpha)\|^2 \leq k\|\psi(\alpha\alpha^*)\| \tag{3.4}$$

for all α in A. It follows that $\|\psi(\alpha)\|^2 \leq k\|\psi(\alpha\alpha^*)\| \leq k^{1+1/2}\|\psi([\alpha\alpha^*]^2)\|^{1/2}$ $\ldots \leq k^2\|\psi\|^{o}\|\hat{\alpha}\|_\infty^2$ and hence, that $\hat{\psi}$ is a bounded linear map.

Since A is self-adjoint and commutative, $\hat{A}$ is dense in $C_o(\mathcal{M})$ and $\hat{\psi}$ can, therefore, be extended to $C_o(\mathcal{M})$. Let $\hat{\psi}_e$ denote the extension of $\hat{\psi}$ to $C_o(\mathcal{M})$. We claim that there is a weak-*-regular set function ν^{**} on $\Sigma(\mathcal{M})$ such that

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$^+$If $\mathcal{M}$ is compact, then $C_o(\mathcal{M})$ is the set of all continuous complex valued functions on $\mathcal{M}$. If $\mathcal{M}$ is locally compact but not compact, then $C_o(\mathcal{M})$ is the set of all continuous complex valued functions on $\mathcal{M}$ which "vanish at infinity".

$$(3.5) \qquad (\hat{\psi}_e(f),\varphi) = \int_{\mathcal{M}} f(\mathcal{M})\,d(\nu^{**},\varphi)$$

for all f in $C_o(\mathcal{M})$ and φ in X^*.

To verify this claim, we let $M(\mathcal{M})$ be the space of all complex valued regular measures μ on $\mathcal{M}$ for which $\|\mu\|$ is finite ([2]). Note that $C_o(\mathcal{M})^* = M(\mathcal{M})$ by the Riesz representation theorem. If $E \in \Sigma(\mathcal{M})$, then let T_E be the element of $C_o(\mathcal{M})^{**}$ defined by

$$(3.6) \qquad T_E(\mu) = \mu(E), \quad \mu \in M(\mathcal{M})$$

Now define a set function ν^{**} of $\Sigma(\mathcal{M})$ into X^{**} by setting

$$(3.7) \qquad \nu^{**}(E) = \hat{\psi}_e^{**}(T_E)$$

for E in $\Sigma(\mathcal{M})$. We show that ν^{**} is weak-*-regular. If φ is an element of X^*, then $\hat{\psi}_e^*(\varphi)$ is, by the Riesz representation theorem, a measure μ_φ in $M(\mathcal{M})$. But

$$(3.8) \qquad \mu_\varphi(E) = T_E(\mu_\varphi) = T_E(\hat{\psi}_e^*(\varphi)) = \hat{\psi}_e^{**}(T_E)(\varphi) = (\nu^{**}(E),\varphi)$$

and so, the set function ν^{**} is weak-*-regular. Moreover, since $\hat{\psi}_e^*(\varphi) = (\nu^{**}(\cdot),\varphi)$ by 3.8, the mapping $T_{\nu^{**}}$ satisfies (iv). Also, $(\hat{\psi}_e(f),\varphi) = \hat{\psi}_e^*(\varphi)(f) = \int_{\mathcal{M}} f(M)\,d\mu_\varphi = \int_{\mathcal{M}} f(M)\,d(\nu^{**},\varphi)$ for f in $C_o(\mathcal{M})$ so that 3.3 is satisfied. It is easy to check that $\|\nu^{**}\|(\mathcal{M}) = \|\hat{\psi}_e\|$ ([7], p. 492) and so, (iii) is satisfied.

All that remains to establish the first half of the theorem is to prove that ν^{**} is Φ-positive. If f is an element of $C_o(\mathcal{M})$ with $f(M) \geq 0$ for all M, then $f^{1/2}$ is in $C_o(\mathcal{M})$ and there is a sequence $\{\alpha_n\}$ in A such that $\lim_{n\to\infty} \hat{\alpha}_n = f^{1/2}$. Since $(\hat{\psi}(\alpha_n\alpha_n^*),\varphi) = \int_{\mathcal{M}} |\hat{\alpha}_n(M)|^2 d(\nu^{**},\varphi)$, it follows that if φ is an element of Φ, then $0 \leq (\psi(\alpha_n\alpha_n^*),\varphi) = \int_{\mathcal{M}} |\hat{\alpha}_n(M)|^2 d(\nu^{**},\varphi)$ and hence, by taking limits, that

$$\int_{\mathcal{M}} f(M)d(\nu^{**},\varphi) \geq 0 \tag{7.9}$$

for all φ in Φ and all f in $C_o(\mathcal{M})$ with $f(\cdot) \geq 0$. But $(\nu^{**}(\cdot),\varphi)$ when restricted to the Baire sets in $\mathcal{M}$ is a Baire measure, and as such , is positive. The Baire measure can be extended to a <u>unique</u> regular Borel measure ([9]) which must (by uniqueness) be $(\nu^{**}(\cdot),\varphi)$. It follows that ν^{**} is Φ-positive.

Now suppose that ν^{**} is given. Since the mapping $T_{\nu^{**}}$ is continuous in the X and $C_o(\mathcal{M})$ topologies, the linear mapping $\varphi \to \int_{\mathcal{M}} f(M)d(\nu^{**},\varphi)$ is, for each fixed f in $C_o(\mathcal{M})$, continuous in the X-topology of X^* and is, therefore, generated by an element x_f of X. Thus, the mapping $\hat{\psi}_e$ of $C_o(\mathcal{M})$ into X given by $\hat{\psi}_e(f) = x_f$ is a bounded linear map of $C_o(\mathcal{M})$ into X. If α is an element of A, then let $\psi(\alpha) = \hat{\psi}_e(\hat{\alpha})$. Since $\|\psi(\alpha)\| = \|\hat{\psi}_e(\hat{\alpha})\| \leq \|\hat{\psi}_e\|\|\hat{\alpha}\|_\infty \leq \|\hat{\psi}_e\|\|\alpha\|$, ψ is a continuous linear map. If φ is an element of Φ, then $(\psi(\alpha\alpha^*),\varphi) = \int_{\mathcal{M}} |\hat{\alpha}(M)|^2 d(\nu^{**},\varphi) \geq 0$ and $(\psi(\alpha^*),\varphi) = \int_{\mathcal{M}} \overline{\hat{\alpha}}(M)d(\nu^{**},\varphi) = \overline{\int_{\mathcal{M}} \hat{\alpha}(M)d(\nu^{**},\varphi)} = \overline{(\psi(\alpha),\varphi)}$ so that ψ is Φ-positive. Also, $|(\psi(\alpha),\varphi)|^2 \leq [\int_{\mathcal{M}} |\hat{\alpha}(M)|^2 d(\nu^{**},\varphi)]$

$[\int_{\mathcal{M}} 1^2 d(\nu^{**},\varphi)] \leq (\psi(\alpha\alpha^*),\varphi)(\nu^{**}(\mathcal{M}),\varphi) \leq \|\nu^{**}\|(\mathcal{M})\|\varphi\|(\psi(\alpha\alpha^*),\varphi) \leq$ $\max\{1,\|\nu^{**}\|(\mathcal{M})/\|\psi\|\}\|\psi\|\|\varphi\|(\psi(\alpha\alpha^*),\varphi)$ so that ψ is almost extendable.

COROLLARY 3.10. *Let A be a self-adjoint commutative Banach algebra with $\widehat{(\alpha^*)} = \overline{\hat{\alpha}}$ and let Φ be a full family. If ψ is Φ-positive and almost extendable and if $\hat{\psi}$ is weakly compact, then there is a weakly regular Φ-positive vector measure ν on $\Sigma(\mathcal{M})$ such that*

$$\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M)d\nu \tag{3.11}$$

for all α in A. Conversely, if ν is a weakly regular Φ-positive vector measure and ψ is given by (3.11), then ψ is Φ-positive and almost extendable and $\hat{\psi}$ is weakly compact.

Proof: Suppose that ψ is given. Since $\hat{\psi}$ is weakly compact, $\hat{\psi}_e$[+] is weakly compact and so, $\hat{\psi}_e^{**}(C_o(\mathcal{M})^{**})$ is contained in the natural imbedding of X in X^{**}. Thus, the set function ν^{**} given by 3.7 may be identified with a mapping ν of $\Sigma(\mathcal{M})$ into X. In that case, $(\nu(\cdot),\varphi)$ is an element of $M(\mathcal{M})$ for all φ in X^*. It follows that $\nu(\cdot)$ is a weakly regular vector measure (as $\nu(\cdot)$ is weakly countably additive). Clearly ν is Φ-positive. Moreover, since $(\psi(\alpha),\varphi) = \int_{\mathcal{M}} \hat{\alpha}(M)d(\nu,\varphi) = (\int_{\mathcal{M}} \hat{\alpha}(M)d\nu,\varphi)$ for all φ in X^*, 3.11 is satisfied.

On the other hand, if ν is given and ψ is defined by 3.11 (note that $\hat{\alpha}(\cdot)$ is bounded and continuous), then ψ is Φ-positive and

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[+]We use the notation of the proof of the theorem.

almost extendable. In fact, $\|\psi(\alpha)\| \leq \|\hat{\alpha}\|_{\infty}\|\nu\|(\mathscr{M}) \leq \|\alpha\|\|\nu\|(\mathscr{M})$ and $|(\psi(\alpha),\varphi)|^2 \leq (\psi(\alpha\alpha^*),\varphi)(\nu(\mathscr{M}),\varphi)$ so that ψ is extendable ($x_o = \nu(\mathscr{M}) \in X$). Thus, to complete the proof we need only show that $\hat{\psi}$ is weakly compact.

Now, $\hat{\psi}$ is clearly linear and, since $\|\hat{\psi}(\hat{\alpha})\| = \|\psi(\alpha)\| \leq \|\nu\|(\mathscr{M})\|\hat{\alpha}\|_{\infty}$, $\hat{\psi}$ is continuous. Let $\hat{\psi}_e$ be the mapping of $C_o(\mathscr{M})$ into X defined by $\hat{\psi}_e(f) = \int_{\mathscr{M}} f(M)d\nu$. Thus, it will be enough to prove that $\hat{\psi}_e$ is weakly compact.

If φ is an element of X^*, then $\hat{\psi}^*_e(\varphi) = (\nu(\cdot),\varphi)$ is an element of $M(\mathscr{M})$. Since the set $\{(\nu(\cdot),\varphi): \varphi \in X^*, \|\varphi\| \leq 1\}$ is weakly sequentially compact as a subset of the space of scalar measures and since ν is weakly regular, $\hat{\psi}^*_e$ is a weakly compact mapping. It follows that $\hat{\psi}_e$ is weakly compact and the corollary is established.

COROLLARY 3.12. _If_ ν _satisfies the conditions of corollary_ 3.10 _and_ ψ _is given by_ 3.11, _then_ ψ _is extendable. Conversely, if_ ψ _is extendable (rather than almost extendable) and if the involution on_ A _is continuous (e.g._ A _semi-simple), then a_ ν _satisfying the conditions of corollary_ 3.10 _exists (the other hypotheses of corollary_ 3.10 _are, of course, assumed)._

Proof: The first assertion was established in the course of the proof of corollary 3.10. The second assertion is an immediate consequence of corollary 2.14.

COROLLARY 3.13. _If_ X _is weakly complete, if_ A _and_ Φ _satisfy the conditions of corollary_ 3.10, _and if_ ψ _is_ Φ_-positive and almost extendable, then_ $\hat{\psi}$ _is weakly compact._

Proof: By the argument given in the proof of theorem 3.2, $\hat{\psi}$ is a continuous linear map. If A has a unite, then $\mathcal{M}$ is compact. Since $\hat{A}$ is dense in $C_o(\mathcal{M})$, we may extend $\hat{\psi}$ to a continuous linear map $\hat{\psi}_e$ of $C_o(\mathcal{M})$ into X. As X is weakly complete, $\hat{\psi}_e$ is weakly compact ([7], p. 494) and a fortiori so is $\hat{\psi}$. If A does not have a unit, then we extend A to $\tilde{A} = A \oplus C$. Letting x_o be an element of X, we extend ψ to a mapping $\tilde{\psi}$ of $\tilde{A}$ into X by setting $\tilde{\psi}(\alpha+\lambda e) = \psi(\alpha)+\lambda x_o$. Then $\hat{\tilde{\psi}}(\hat{\alpha}+\lambda\hat{e}) = \hat{\psi}(\hat{\alpha})+\lambda x_o$ is a bounded linear map of $\hat{\tilde{A}}$ into X. It follows that $\hat{\tilde{\psi}}$ is weakly compact and hence, that $\hat{\psi}$ is weakly compact.

4. Bochner's Theorem on a Group

Let G be a σ-finite locally compact abelian group and let $A = L_1(G,C)$. The involution on A is given by $\alpha^*(g) = \overline{\alpha(-g)}$ and is continuous. Let X be a Banach space and let Φ be a full family. We shall prove a generalization of Bochner's theorem for integrally Φ-positive definite mappings p in $L_\infty(G,X)$ by combining lemma 2.20 with theorem 3.2 and its corollaries. We have

THEOREM 4.1. (A) *If ν is a weakly regular Φ-positive vector measure defined on $\Sigma(\hat{G})$ (the Borel field of the dual group $\hat{G}$) and if*

$$p(g) = \int_{\hat{G}} \overline{(\gamma, g)}\, d\nu \tag{4.2}$$

then p is an integrally Φ-positive definite element of $L_\infty(G,X)$.

(B) *If p is an integrally Φ-positive definite element of $L_\infty(G,X)$, then there is a set function ν^{**} mapping $\Sigma(\hat{G})$ into X^{**} such* *that* (i) ν^{**} *is weak-*-regular,* Φ*-positive, and finite,* (ii) *the map* $T_{\nu^{**}}$ *given by* $T_{\nu^{**}}(\varphi) = (\nu^{**}(\cdot),\varphi)$ *is continuous in the* X *topology of* X^* *and the* $C_o(\hat{G})$ *topology of* $M(\hat{G})$, *and* (iii)

$$(4.3) \qquad (p(g),\varphi) = \int_{\hat{G}} \overline{(\gamma, g)}\, d(\nu^{**},\varphi)$$

for all φ *in* X^* *and (almost) all* g *in* G.

Proof: (A) Let $p(\cdot)$ be given by 4.2. Suppose, for the moment, that $p(\cdot)$ is measurable. Then p is in $L_\infty(G,X)$ since $\|p(g)\| \leq \|\nu\|(\hat{G})$ for all g. Let $\psi(\alpha) = \int_G \alpha(g)p(g)d\mu$ for α in $L_1(G,C)$. Then

$$(\psi(\alpha),\varphi) = \int_G \alpha(g) \int_{\hat{G}} \overline{(\gamma, g)}\, d(\nu,\varphi)d\mu$$

$$= \int_{\hat{G}} \int_G \alpha(g) \overline{(\gamma, g)}\, d\mu d(\nu,\varphi)$$

$$= \int_{\hat{G}} \hat{\alpha}(\gamma) d(\nu,\varphi) = \left(\int_{\hat{G}} \hat{\alpha}(\gamma) d\nu,\varphi\right)$$

for all φ in X^* by the Fubini and Tonelli theorems. Since $\hat{G}$ and $\mathscr{M}$ can be identified ([2] or [3]), we have $\psi(\alpha) = \int_{\mathscr{M}} \hat{\alpha}(M)d\nu$ (as ν may be viewed as a measure on $\mathscr{M}$). But then (corollary 3.10) ψ is Φ-positive and extendable (corollary 3.12). The result follows immediately from 2.22 of lemma 2.20.

Thus, to complete the proof of (A), we need only show that p is measurable. To do this it will be sufficient to show that, for any set $F \subset G$ with $\mu(F) < \infty$, $P_F(\cdot) = \chi_F(\cdot)p(\cdot)$ is the limit in measure of a sequence of simple functions where χ_F is the characteristic function of F.

Since ν is weakly regular, there is a finite, positive, regular scalar measure λ such that $\|\nu\|(E) \to 0$ if and only if $\lambda(E) \to 0$ where $\|\nu\|(E)$ is the semi-variation of ν on E ([7]). Therefore, given $\eta > 0$, there is a $\xi > 0$ such that if $\lambda(\hat{G}-K)^+ < \xi$, then $\|\nu\|(\hat{G}-K) < \eta/4$ for K compact in $\hat{G}$. Since λ is finite and regular, there is a compact set $K \subset \hat{G}$ for which $\lambda(\hat{G}-K) < \xi$ and hence for which $\|\nu\|(\hat{G}-K) < \eta/4$. Let $\eta_1 = \eta/2\|\nu\|(\hat{G})$ and let $N(g;K,\eta_1) = \{g' \in G\colon |1-(\gamma,g')| < \eta_1,\ \gamma \in K\}+g$. Then $N(g;K,\eta_1)$ is an open neighborhood of g in G.

Now G is σ-finite and so there is an increasing sequence of sets G_n with $\mu(G_n) < \infty$ and $\cup G_n = G$. Moreover, since Haar measure is regular, given $\epsilon > 0$ there is a compact set $L_n \subseteq G_n$ such that $\mu(G_n-L_n) < \epsilon$. The sets $N(g;K,\eta_1)$, $g \in L_n$, form an open cover of L_n. Thus there are $g_1,\ldots,g_{M_n}$ in L_n such that $L_n \subseteq \bigcup_{i=1}^{M_n} N(g_i;K,\eta_1)$. Let $N_1^n = N(g_1;K,\eta_1)$ and $N_{i+1}^n = N(g_{i+1};K,\eta_1) - (N(g_1;K,\eta_1) \cup \ldots \cup N(g_i;K,\eta_1))$. Then $L_n \subseteq \bigcup_{i=1}^{M_n} N_i^n$ and the union is disjoint. Let p_o be defined on L_n by $p_o(g) = p(g_i)$ if $g \in N_i^n$ and let $p_n^{\epsilon,\eta}(\cdot)$ be given by

$$(4.4)\qquad p_n^{\epsilon,\eta}(g) = \begin{cases} p_o(g) & g \in L_n \\ 0 & g \notin L_n \end{cases}$$

Then $p_n^{\epsilon,\eta}(\cdot)$ is a simple function and we claim that

$$(4.5)\qquad \mu^*(\{g \in G_n\colon \|p(g)-p_n^{\epsilon,\eta}(g)\| > \eta\}) < \epsilon$$

where $\mu^*(E) = \inf_{E_1} \mu(E_1)$, $E_1 \supseteq E$. For if g is in L_n, then $\|p(g)-{}_1^{\epsilon,\eta}(g)\| =$

$^+$Here $\hat{G}-K$ is the complement of K.

$\|p(g)-p_o(g)\| = \|\int_{\hat{G}}\overline{(\gamma,g)}[1-\overline{(\gamma,g_i-g)}]d\nu\| \leq \|\int_{\hat{G}-K}\overline{(\gamma,g)}[1-\overline{(\gamma,g_i-g)}]d\nu\| +$ $\|\int_K\overline{(\gamma,g)}[1-\overline{(\gamma,g_i-g)}]d\nu\| \leq \frac{\eta}{2} + \eta_1\|\nu\|(\hat{G}) = \eta$ (for some i) so that $\{g \in G_n:$ $\|p(g)-p_n^{\epsilon,\eta}(g)\| > \eta\} \subseteq G_n-L_n$. It follows that 4.5 holds. Let $p_n(g) =$ $p_n^{1/n,1/n}(g)$ so that p_n is a simple function.

Now suppose that a is any positive number. We show that

$$\lim_{n\to\infty} \mu^*(\{g \in F: \|p(g) - p_n(g)\| > a\}) = 0 \tag{4.6}$$

for any $F \subset G$ with $\mu(F) < \infty$. So let $\epsilon > 0$ be given. Then there is an $n_o \geq \max(1/a, 2/\epsilon)$ such that $\mu(F \cap (G-G_n)) < \epsilon/2$ for $n \geq n_o$. It follows that

$$\begin{aligned}\mu^*(\{g\epsilon F: \|p(g)-p_n(g)\| > a\}) &\leq \mu^*(\{g\epsilon F \cap G_n, \|p(g)-p_n(g)\| > a\}) + \epsilon/2\\ &\leq \mu^*(\{g\epsilon F \cap G_n: \|p(g)-p_n(g)\| > 1/n\}) + \epsilon/2\\ &\leq 1/n + \epsilon/2 \leq \epsilon\end{aligned}$$

for $n \geq n_o$. In other words, p_n converges to p in measure on F. The proof of (A) is now complete.

(B) Let $\psi(\alpha) = \int_G \alpha(g)p(g)d\mu$. Then ψ is Φ-positive and almost extendable by lemma 2.20. It follows from theorem 3.2 that there is a set function ν^{**} on $\Sigma(\mathcal{M})$ such that (i) and (ii) are satisfied and

$$(\psi(\alpha),\varphi) = \int_{\mathcal{M}} \hat{\alpha}(M)d(\nu^{**},\varphi) \tag{4.7}$$

for all φ in X^*. Since $\hat{G}$ and $\mathcal{M}$ can be identified, ν^{**} may be viewed as a set function on $\Sigma(\hat{G})$ and

$$(4.8) \qquad (\psi(\alpha),\varphi) = \int_{\hat{G}}[\int_G \alpha(g)\overline{(\gamma,g)}d\mu]d(\nu^{**},\varphi)$$

for all φ in X^*. Application of the Fubini and Tonelli theorems then yields

$$(4.9) \qquad \int_G \alpha(g)(p(g),\varphi)d\mu = (\psi(\alpha),\varphi) = \int_G \alpha(g)q_\varphi(g)d\mu$$

where $q_\varphi(g) = \int_{\hat{G}}\overline{(\gamma,g)}d(\nu^{**},\varphi)$. Since $(p(\cdot),\varphi)$ and $q_\varphi(\cdot)$ are in $L_\infty(G,C)$, we have $\|(p(\cdot),\varphi)-q_\varphi(\cdot)\|_\infty = 0$ for all φ in X^*. In other words, 4.3 holds. The proof of (B) is now complete.

REMARK 4.10. *Since* $\overline{(\gamma,g)} = (-\gamma,g)$ *and since the measure* ν_1 (*or the set function* ν_1^{**}) *given by* $\nu_1(E) = \nu(-E)$ (*or* $\nu_1^{**}(E) = \nu^{**}(-E)$) *has the same properties as* ν (*or* ν^{**}), $p(g)$ *is given by* $p(g) = \int_{\hat{G}}(\gamma,g)d\nu_1$ (*or satisfies* $(p(g),\varphi) = \int_{\hat{G}}(\gamma,g)d(\nu_1^{**},\varphi)$). [*This agrees with convention in the scalar case.*]

We observe that if the hypotheses of (A) are satisfied and $\psi(\alpha) = \int_G \alpha(g)p(g)d\mu$, then the mapping $\hat{\psi}$ of $\hat{A}$ into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$ is weakly compact (corollary 3.10). Note also that if X is weakly complete and $p(\cdot)$ is an integrally Φ-positive definite element of $L_\infty(G,X)$, then $\hat{\psi}$ is weakly compact. This leads to

COROLLARY 4.11. *If* X *is weakly complete and if* p *is an integrally* Φ-

positive definite element of $L_\infty(G,X)$, then there is a weakly regular Φ-positive vector measure ν on $\Sigma(\hat{G})$ such that

$$(4.12) \qquad (p(g),\varphi) = (\int_{\hat{G}}(\gamma,g)d\nu,\varphi)$$

for all φ in X^* and (almost) all g in G. If, in addition, Φ is countable, then

$$(4.13) \qquad p(g) = \int_{\hat{G}}(\gamma,g)d\nu$$

for (almost) all g in G.

Proof. The first assertion follows from corollary 3.12. On the other hand, if $\Phi = \{\varphi_i\}$ is countable, then there is a μ-null set N such that $(p(g)-q(g),\varphi) = 0$ for all φ in Φ and $g \notin N$ where $q(g) = \int_{\hat{G}}(\gamma,g)d\nu$. But then $\|p(g)-q(g)\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(p(g)-q(g),\varphi)|/\|\varphi\|\} = 0$ for $g \notin N$. It follows immediately that $\|p(\cdot)-q(\cdot)\|_\infty = 0$, i.e. that 4.13 holds.

In order to state our final corollary we require

DEFINITION 4.14. The element p of $L_\infty(G,X)$ is dominated if there exists a finite regular positive Borel measure λ such that

$$(4.15) \qquad \|\int_G \alpha(g)p(g)d\mu\| \leq \int_{\hat{G}}|\hat{\alpha}(\gamma)|d\lambda$$

for all α in $L_1(G,C)$, where $\hat{\alpha}$ is the Fourier transform of α.

COROLLARY 4.16. *Assume Φ is countable. Then p is a dominated integrally Φ-positive definite element of $L_\infty(G,X)$ if and only if there exists a weakly regular Φ-positive vector measure ν of finite variation mapping $\Sigma(\hat{G})$ into X such that*

$$(4.17) \qquad p(g) = \int_{\hat{G}}(\gamma,g)d\nu.$$

Proof: We have only to note that there exists an isomorphism between the set of weakly regular vector measures $\nu: \Sigma(\hat{G}) \to X$ with finite variation and the set of bounded linear operators $T: C_o(\hat{G}) \to X$ for which there exists a finite regular positive Borel measure λ such that $\|T(f)\| \leq \int_{\hat{G}}|f(\gamma)|d\lambda$. This isomorphism is given by $T(f) = \int_G f(\gamma)d\nu$, ([9], p. 380, or [11]). Now using theorem 4.1 (B) we have, if we assume the existence of p, that $(\int_{\hat{G}}f(\gamma)d\nu,\varphi) = \int_{\hat{G}}f(\gamma)d(\nu^{**},\varphi)$ for any f in $C_o(\hat{G})$, φ in X^*. But $C_o(\hat{G})^* = M(\hat{G})$, the space of regular complex valued measures defined on $\Sigma(\hat{G})$ of finite variation, and (φ,ν^{**}), (ν,φ) are in $M(\hat{G})$. Thus, for any E in $\Sigma(\hat{G})$, $(\nu(E),\varphi) = (\varphi,\nu^{**}(E))$. Consider $\nu(E)$ as an element of X^{**}, then $\nu(E) = \nu^{**}(E)$ and so ν^{**} is actually a measure. From the countability of Φ we derive (4.17).

The converse follows immediately from theorem 4.1 (A).

5. Some Examples

We now give several examples of spaces to which the theory applies.

EXAMPLE 5.1. Let $X = L_1([0,1],C)$. Note that X is weakly complete. If $\Sigma = \Sigma([0,1])$ is the Borel field on $[0,1]$, then Σ is a separable metric space with respect to the usual metric $d(E,E') = \mu(E \triangle E')$ where $E \triangle E' =$

$(E-E') \cup (E'-E)$ is the symmetric difference of E and E'. Let $\{E_i\}$ be a countable dense set in Σ with $E_1 = [0,1]$. Let χ_i be the characteristic function of E_i and let φ_i be the element of X^* given by

$$(5.2) \qquad (x(\cdot),\varphi_i) = \int_0^1 \chi_i(s)x(s)ds$$

If $\Phi = \{\varphi_i\}$, then K_Φ is the cone of (essentially) nonnegative functions. Note also that $\|\varphi_i\| \leq 1$.

Now we claim that Φ is full. Set $x^+(s) = \max\{0,x(s)\}$ and $x^-(s) = \max\{0,-x(s)\}$ for real x in X. Then $x(s) = x^+(s) - x^-(s)$ and $|x(s)| = x^+(s) + x^-(s)$. Moreover, x^+ and x^- are nonnegative. Letting $\int_0^1 x_0(s)ds = \max\{\int_0^1 x^+(s)ds, \int_0^1 x^-(s)ds\}$ (i.e. $x_0 = x^+$ or x^- according to which integral is greater), we see that

$$(5.3) \qquad \|x(\cdot)\|_1 \leq 2\int_0^1 x_0(s)ds$$

for real x in X. Now, suppose, for example, that $x_0 = x^+$. Since x^+ is measurable, $(x^+)^{-1}([0,\infty)) = E$ is in Σ and $\int_0^1 x_0(s)ds = \int_E x^+(s)ds = \int_E x(s)ds$. As $\{E_i\}$ is dense in Σ, there is a sequence $\{E_{i,n}\}$ such that $d(E_{i,n},E) \to 0$ as $n \to \infty$. But $|\int_{E_{i,n}} x(s)ds - \int_E x(s)ds| \leq \int_{E_{i,n} \triangle E}|x(s)|ds$ and $\lim_{n\to\infty} \int_{E_{i,n} \triangle E}|x(s)|ds = 0$ as $\mu(E_{i,n} \triangle E) \to 0$ as $n \to \infty$ and x is in $L_1([0,1],R)$. It follows that there is a sequence $\{\varphi_{i,n}\}$ such that $\lim_{n\to\infty}(x,\varphi_{i,n}) = \int_0^1 x_0(s)ds$ and hence, that $\int_0^1 x_0(s)ds \leq \sup_\varphi |(x,\varphi)|$. Now choose any x in X. Then $x = x_1 + ix_2$ where $x_1(\cdot)$, $x_2(\cdot)$ are real valued. But $\|\varphi\| \leq 1$ for φ in Φ and so,

$$(5.4) \qquad \|x\|_1 \leq \|x_1\|_1 + \|x_2\|_1 \leq 4 \sup_{\varphi\in\Phi}|(x,\varphi)| \leq 4 \sup_{\substack{\varphi\in\Phi \\ \varphi\neq 0}}(|(x,\varphi)|/\|\varphi\|)$$

for all x in X.

EXAMPLE 5.5. Let $X = H$ be a separable Hilbert space. Fix an orthonormal basis $\{e_i\}$ in H. If $h \in H$, then $h = h_1+ih_2$ where $h_1 = \sum \mathrm{Re}\{< h,e_i >\}e_i$ and $h_2 = \sum \mathrm{Im}\{ < h,e_i >\}e_i$. An element h is real if $h = h_1$. Let H_o be the set of all real elements h such that (i) $\|h\| \leq 1$, (ii) h is positive i.e. $< h,e_i > \geq 0$ for all i, (iii) h is rational i.e. $< h,e_i >$ is rational for all i, and (iv) h is finite i.e. only a finite number of components $< h,e_i >$ of h are not zero. Since H^* can be identified with H, we let $\Phi = H_o$. In other words, if $\varphi \in \Phi$, then $(h,\varphi) = < h,k >$ for some k in H_o. The cone K_Φ is the set of all positive real elements of H.

We claim that Φ is full. Suppose first that $h = h_1$ is real. Then $h_1 = h_1^+ - h_1^-$ where $< h_1^+,e_i > = \max\{0, < h_1,e_i > \}$ and $< h_1^-,e_i > = \max\{0,- < h_1,e_i > \}$ for all i. Note that $\|h_1\|^2 = \|h_1^+\|^2 + \|h_1^-\|^2$. Let k_n^+ be the element of H_o with components $< k_n^+,e_i >$ given by

$$(5.6) \qquad < k_n^+,e_i > = \begin{cases} r_i & i \leq N \\ 0 & i > N \end{cases}$$

where N is chosen so that

$$(5.7) \qquad \sum_{N+1}^{\infty} |< h_1^+,e_i >|^2 < \frac{1}{2n^2} \|h_1^+\|^2$$

and r_i is a nonnegative rational such that

(5.8) $$< h_1^+, e_i > \geq r_i \|h_1^+\| > \left\{ < h_1^+, e_i > - \frac{\|h_1^+\|}{n\sqrt{2N}} \right\}$$

Clearly $\|k_n^+ - \frac{h_1^+}{\|h_1^+\|}\| < 1/n$. It follows that $(h_1^+, k_n^+) \to \|h_1^+\|$ as $n \to \infty$. Similarly, there is a sequence k_n^- in H_o such that $(h_1^-, k_n^-) \to \|h_1^-\|$ as $n \to \infty$. Noting that for any $h = h_1 + ih_2$ in H, $|(h,\varphi)| \geq \max\{|(h_1,\varphi)|, |(h_2,\varphi)|\}$, we have $\|h_1\|^2 = \|h_1^+\|^2 + \|h_1^-\|^2 = \lim\limits_{n\to\infty} (h_1^+, k_n^+)^2 + \lim\limits_{n\to\infty} (h_1^-, k_n^-)^2 \leq \overline{\lim} |(h_1^+, k_n^+)|^2 + \overline{\lim} |(h_1^-, k_n^-)|^2 \leq 2 \sup\limits_{\varphi \in \Phi} |(h_1,\varphi)|^2$. Now, if $h = h_1 + ih_2$ is any element of H, then $\|h\|^2 = \|h_1\|^2 + \|h_2\|^2 \leq 2 \sup\limits_{\varphi} |(h_1,\varphi)|^2 + 2 \sup\limits_{\varphi} |(h_2,\varphi)|^2 \leq 4 \sup\limits_{\varphi} |(h,\varphi)|^2$. Since $\|\varphi\| \leq 1$ if $\varphi \in \Phi = H_o$, we deduce that $\|h\| \leq 2 \sup\limits_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(h,\varphi)|/\|\varphi\|\}$ for all h in H. Thus, Φ is full.

EXAMPLE 5.9. Let H be a separable Hilbert space and let $X = \mathcal{L}(H,H)$ be the space of bounded linear maps of H into itself. Let H_o be a countable dense subset of the closed unit ball in H and let $\Phi = \{\varphi \in X^*: (T,\varphi) = < Tk,k >$, for some k in $H_o\}$. The cone K_Φ is the set of positive operators in $\mathcal{L}(H,H)$. Since $\|T\| \leq 2 \sup\limits_{\|h\| \leq 1} | < Th,h > |$ for T in $\mathcal{L}(H,H)$ and since $\|\varphi\| \leq \|k\|^2 \leq 1$ for k in H_o, we have $\|T\| \leq 2 \sup\limits_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(T,\varphi)|/\|\varphi\|\}$. In other words, Φ is full.

EXAMPLE 5.10. Let $\mathcal{D}$ be a bounded domain in R^n and let $X = L_p(\mathcal{D}, C)$ where $1 < p < \infty$. Let Σ be the Borel field of $\mathcal{D}$. Then Σ is a separable metric space with respect to the usual metric $d(E,E') = \mu(E \triangle E')$. Let $\Sigma_0 = \{E_i\}$ be a countable dense set in Σ which include all hypercubes with rational vertices contained in $\mathcal{D}$ and let $Q = \{a+bi \in C: a,b$ rational$\}$. Let $\mathcal{S}$ be the set of simple functions of the form

$\sum_{i=1}^{n} q_i \chi_{E_i}$ where the q_i are in Q and the E_i are disjoint elements of Σ_0. Note that $\mathscr{S}$ is a countable subset of $L_q(\mathscr{D},C)$ where $1/p + 1/q = 1$. It is easy to check that $\mathscr{S}$ is dense in $L_q(\mathscr{D},C)$. An element $\sum_{i=1}^{n} q_i \chi_{E_i}$ of $\mathscr{S}$ is positive real if q_i is a nonnegative real number for $i = 1,\ldots,n$. Let Φ be the subset of $\mathscr{S}$ consisting of all the positive real elements. Since $X^* = L_p(\mathscr{D},C)^* = L_q(\mathscr{D},C)$, $\Phi \subset X^*$ and the cone K_Φ is simply the set of nonnegative functions in $L_p(\mathscr{D},C)$. The proof that Φ is full is straightforward and is, therefore, left to the reader. Theorem 4.1, when interpreted in this context, becomes:

COROLLARY 5.11. *If p is an element of $L_\infty(G,L_p(\mathscr{D},C))$ such that $\int_G\int_G \xi(g)\overline{\xi(g')}p(g-g')d\mu d\mu$ is a nonnegative function in $L_p(\mathscr{D},C)$ for all $\xi(\cdot)$ in $L_1(G,C)$, then $p(g) = \int_{\hat{G}}(\gamma,g)d\nu$ where ν is a weakly regular measure on $\hat{G}$ such that $\nu(F)$ is a nonnegative function in $L_p(\mathscr{D},C)$ for all F in $\Sigma(\hat{G})$, and conversely.*

This corollary plays a role in the study of positive solutions of certain partial differential equations.

EXAMPLE 5.12. Let H be a separable Hilbert space and let $\mathscr{C} = \mathscr{C}(H,H)$ be the closed ideal of compact operators in $\mathcal{L}(H,H)$. It is well-known ([8]) that $\mathcal{L}(H,H)^* = \mathcal{L}_1 \oplus \mathscr{C}^\perp$ where $\mathscr{C}^\perp$ is the annihilator of $\mathscr{C}$ and $\mathcal{L}_1$ is the trace class. Moreover, $\mathcal{L}_1$ is isometrically isomorphic with $\mathscr{C}^*$ and $\mathscr{C}^{**}$ is isometrically isomorphic with $\mathcal{L}_1^* = \mathcal{L}(H,H)$. Now let H_0 be a countable dense subset of the closed unit ball in H and let $\Phi = \{\varphi \in \mathscr{C}^*: (T,\varphi) = \langle Tk,k\rangle$ some k in $H_0\}$. The cone K_Φ is the set of positive compact operators and Φ is a countable full family.

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